



ROTATION ANALYTICS

Biomathematical Fatigue Modelling: Technical White Paper

A Technical White Paper on the Biomathematical Fatigue Model
Used by Rotation Analytics

Version 1.1 • March 2026
Confidential

Table of Contents

1. Introduction
2. Theoretical Foundation
3. Model Architecture
4. Model Parameters
5. Fatigue Score Calculation
6. Fatigue Risk Classification
7. Days-in-Risk-Band Analysis
8. Model Assumptions and Limitations
9. Scientific References
10. Document Control

1. Introduction

Fatigue is a primary contributor to workplace incidents in shift work industries. Unlike simple hour-counting approaches, biomathematical fatigue modelling predicts worker alertness by simulating the underlying physiological processes that govern sleep, wakefulness, and cognitive performance.

This white paper describes the biomathematical model used by Rotation Analytics to assess fatigue risk across roster and rotation designs. The model is based on the Three-Process Model (TPM) of alertness regulation, originally developed by Åkerstedt and Folkard (1987–2008) and widely adopted in occupational fatigue research.

The model produces a Fatigue Score on a 0–100 scale for every day of the rotation for every rotation line. Scores are classified into five risk levels (Low, Moderate, High, Very High, and Critical), enabling stakeholders to identify schedule segments associated with elevated fatigue exposure and make informed decisions about rotation design.

2. Theoretical Foundation

2.1 The Three-Process Model

The TPM describes alertness as the interaction of three independent physiological processes:

- **Process S — Homeostatic Sleep Pressure:** A drive to sleep that builds exponentially during wakefulness and dissipates exponentially during sleep. The longer a person is awake, the stronger the pressure to sleep; the longer they sleep, the more refreshed they become.
- **Process C — Circadian Rhythm:** A ~24-hour biological oscillation governed by the suprachiasmatic nucleus, which modulates alertness independently of sleep history. Alertness peaks in the late afternoon (~16:48) and reaches its trough in the early morning (~04:48).
- **Process W — Sleep Inertia:** A transient reduction in performance immediately upon waking. This process is omitted from the current model as it is designed for master schedule analysis rather than short-notice call-in assessment. Sleep inertia effects are typically confined to the first 30–60 minutes after awakening and are not material to rotation-level fatigue risk assessment.

The model produces a Fatigue Score on a 0–100 scale, where higher values indicate greater fatigue and degraded cognitive performance.

3. Model Architecture

3.1 Simulation Approach

The model operates as a time-stepping simulation with 15-minute resolution ($\Delta t = 0.25$ hours). For each rotation line, the model:

- Reads the complete shift schedule from the rotation design
- Simulates the full rotation cycle twice (the first pass establishes steady-state conditions; the second pass produces scored results)
- At each 15-minute step, updates the worker's physiological state based on whether they are sleeping, awake off-duty, or working
- Calculates a composite Fatigue Score for each worked shift

The cyclic warm-up ensures that the model's initial conditions reflect the true steady-state of the repeating rotation, eliminating transient artefacts from arbitrary starting values.

3.2 Sleep Prediction

Rather than assuming fixed sleep windows, the model simulates sleep onset and offset using threshold-crossing mechanics:

- Sleep onset occurs when the worker's predicted alertness falls below a threshold (0.55), indicating sufficient physiological sleep drive
- Waking occurs when alertness recovers above a higher threshold (0.85), simulating natural sleep termination
- A circadian sleep gate between 22:00 and 06:00 lowers the effective onset threshold by 0.20, reflecting the well-documented facilitation of nighttime sleep
- Daytime sleep is capped at 5.5 hours per calendar day, consistent with published research showing that night-shift workers obtain significantly less restorative sleep when sleeping during biological daytime (Åkerstedt, 2003; Folkard & Tucker, 2003)
- An overall cap of 10 hours per sleep opportunity prevents unrealistic extended sleep episodes

This approach naturally captures key phenomena observed in shift workers:

- Difficulty sleeping during the day after night shifts
- Shortened sleep episodes during quick turnarounds
- Improved sleep quality during nighttime rest periods
- Progressive sleep restriction across consecutive night shifts

3.3 First-Night-in-Run Enhancement

The model includes specific handling for the first night shift following a block of day shifts or days off. Research consistently identifies this transitional shift as carrying the highest fatigue risk, as workers have typically been awake since their normal morning wake time and have not adapted their sleep schedule to night work.

When a night shift is detected as the first in a run (i.e., the previous worked shift was not a night shift), the model suppresses all sleep from 06:00 until the shift start time. This reflects the realistic scenario where workers on a daytime schedule do not nap before their first night. They wake at their usual time and remain awake through to the shift. For example: a 19:00 shift start would produce approximately 13

hours of pre-shift wakefulness; a 23:00 start, approximately 17 hours.

This enhancement ensures that first-night fatigue scores exceed those of subsequent nights in the same block, consistent with occupational fatigue literature (Folkard & Tucker, 2003; Åkerstedt, 2003).

4. Model Parameters

4.1 Homeostatic Process (S)

Parameter	Value	Description
τ_W (tauW)	18.2 hours	Time constant for S buildup during wakefulness
τ_S (tauS)	4.2 hours	Time constant for S decay during sleep (initial rapid recovery)
τ_{S2} (tauS2)	8.4 hours	Time constant for S decay during deep sleep (slower secondary recovery)
S_lower	0.17	Asymptotic lower bound of S (fully rested state)
S_brake	0.35	Threshold at which sleep recovery transitions from rapid (τ_S) to slow (τ_{S2})

Table 1: Homeostatic process parameters.

The two-speed decay model reflects real sleep architecture: initial sleep produces rapid recovery of alertness, while extended sleep yields diminishing returns. The transition at S_brake approximates the shift from deep slow-wave sleep to lighter sleep stages.

During wakefulness:

$$S(t+dt) = 1 - (1 - S(t)) \times \exp(-dt / \tau_W)$$

During sleep ($S > S_{\text{brake}}$):

$$S(t+dt) = S_{\text{lower}} + (S(t) - S_{\text{lower}}) \times \exp(-dt / \tau_S)$$

During sleep ($S \leq S_{\text{brake}}$):

$$S(t+dt) = S_{\text{lower}} + (S(t) - S_{\text{lower}}) \times \exp(-dt / \tau_{S2})$$

4.2 Circadian Process (C)

Parameter	Value	Description
Amplitude (ampC)	0.40	Amplitude of circadian oscillation (published TPM range: 0.4–0.5)
Acrophase (ϕ)	16.8 hours	Time of peak alertness (~16:48, late afternoon)

Table 2: Circadian process parameters.

The circadian process is modelled as a cosine function:

$$C(t) = \text{ampC} \times \cos(2\pi(t - \phi) / 24)$$

At amplitude 0.40, the circadian signal produces meaningful physiological effects:

- **Peak alertness (+0.40)** at ~16:48. Strongly promotes wakefulness, blocks daytime sleep in well-rested individuals.
- **Trough alertness (-0.40)** at ~04:48. Strongly promotes sleep onset, contributes to the well-documented circadian nadir during early-morning hours.
- The amplitude is set within the range validated in published TPM literature (Åkerstedt et al., 2004; Folkard et al., 2005).

4.3 Sleep Prediction Parameters

Parameter	Value	Description
Sleep onset threshold	0.55	Alertness level below which sleep initiation occurs
Wake threshold	0.85	Alertness level above which spontaneous waking occurs
Sleep gate bonus	0.20	Additional threshold allowance during 22:00–06:00
Daytime sleep cap	5.5 hours/day	Maximum daytime (06:00–18:00) sleep per calendar day
Maximum sleep	10 hours	Overall cap per sleep opportunity
Required sleep	8 hours	Baseline daily sleep need for sleep debt calculation

Table 3: Sleep prediction parameters.

4.4 Scoring Parameters

Parameter	Value	Description
Circadian weight	25	Scaling factor converting raw circadian value to fatigue points
Sleep debt rate (k_debt)	0.5	Fatigue points added per hour of accumulated sleep debt
Debt recovery rate	0.5	Rate at which excess sleep reduces accumulated debt (50% efficiency)

Table 4: Scoring parameters.

5. Fatigue Score Calculation

For each worked shift, the Fatigue Score is computed from three components:

5.1 Sleep Pressure Component (FatigueBase)

The model tracks S continuously through the shift. The peak value of S during the shift is used as the basis:

$$\text{FatigueBase} = S_{\text{peak}} \times 100$$

Using peak S reflects the worst-case fatigue exposure during the shift, which is the most safety-relevant metric. For a 12-hour shift, this typically occurs at the end of the shift when cumulative wakefulness is greatest.

5.2 Circadian Component (CircadianMod)

The average circadian value across all 15-minute steps of the shift is computed:

$$\text{CircadianMod} = -C_{\text{avg_shift}} \times \text{CircadianWeight}$$

When the circadian function is negative (nighttime trough), CircadianMod becomes positive, adding to fatigue. When the circadian function is positive (daytime peak), CircadianMod becomes negative, reducing fatigue.

Typical circadian contributions by shift window:

Shift Window	Avg C	CircadianMod	Effect
Day 07:00–19:15	+0.14	–3.5 pts	Reduces fatigue
Night 19:00–07:15	–0.14	+3.5 pts	Increases fatigue
Day 07:00–15:15	+0.02	–0.5 pts	Mild reduction
Evening 15:00–23:15	+0.26	–6.6 pts	Moderate reduction
Night 23:00–07:15	–0.29	+7.3 pts	Significant increase

Table 5: Circadian modifier by shift window.

5.3 Sleep Debt Component (DebtMod)

Cumulative sleep debt is tracked across the rotation. When actual sleep falls short of the 8-hour requirement, the deficit accumulates. When sleep exceeds the requirement, debt recovers at 50% efficiency (reflecting that recovery sleep is less efficient than preventive sleep).

$$\text{DebtMod} = \text{SleepDebt_cumulative} \times k_{\text{debt}}$$

5.4 Composite Score

$$\text{FatigueScore} = \text{FatigueBase} + \text{CircadianMod} + \text{DebtMod}$$

(bounded to 0–100)

Where:

- **FatigueBase** = $S_{\text{peak}} \times 100$, converting the 0–1 sleep pressure value to a 0–100 base score

- **CircadianMod** = $-C_avg_shift \times 25$, ranging from approximately -7 (afternoon, favourable) to $+7$ (early morning, unfavourable)
- **DebtMod** adds 0.5 points per hour of accumulated sleep debt

The composite score is clamped to the 0–100 range. A score of 0 represents minimal physiological fatigue. A score of 100 represents the upper bound of the model's fatigue range. All scores represent a **representative worker**: a healthy working-age adult with no pre-existing sleep disorders, following the scheduled shift pattern without overtime or shift swaps.

6. Fatigue Risk Classification

Fatigue Scores are mapped to five risk categories. These thresholds have been calibrated against expected fatigue outcomes for common shift patterns.

Interpretation guidance. Risk classifications describe the modelled physiological state of a representative worker at a given point in the rotation. They are not compliance determinations, safety guarantees, or predictions of individual worker performance. Classifications are intended to support scheduling review by identifying where the schedule creates conditions associated with elevated fatigue exposure.

Score Range	Classification	Description
0 – 44	Low	Worker is operating within optimal physiological parameters for a standard shift.
45 – 59	Moderate	Slightly elevated fatigue. Typical of evening shifts and consecutive day shifts with mildly shortened sleep. Performance may be subtly degraded. This is the expected baseline result for shift workers.
60 – 74	High	Significant fatigue. Typical of night shifts and extended shifts, where the circadian trough compounds sleep pressure. Meaningful degradation in reaction time and decision-making expected.
75 – 84	Very High	High fatigue consistent with extended 12-hour night shifts and consecutive night shift blocks. The circadian trough and cumulative sleep restriction compound to produce sustained performance impairment.
85 – 100	Critical	Severe fatigue. Occurs with extended wakefulness, deep circadian trough, and compounding sleep debt. Performance impairment is comparable to levels documented in occupational health fatigue literature. Immediate schedule review required.

Table 6: Risk classification framework with calibrated thresholds.

6.1 Calibration Benchmarks

The model has been calibrated against expected fatigue outcomes for common shift patterns:

Shift Pattern	Typical Score Range	Risk Band
Days off / rest periods	38–42	Low
8-hour day shifts (consecutive)	50–54	Moderate
8-hour evening shifts	56–60	Moderate
12-hour day shifts	59–60	Moderate–High
8-hour night shifts	70–75	High
12-hour night shifts	80–83	Very High
3rd consecutive 12-hour night	82–83	Very High

Table 7: Calibration benchmarks for common shift patterns.

7. Days-in-Risk-Band Analysis

In addition to average and peak fatigue scores, the model reports the number of days each worker spends in each risk band across the full rotation cycle. This metric captures the cumulative exposure profile. Two rotations may have similar average scores but very different distributions of high-risk days.

This analysis includes all days of the rotation (work days and rest days), providing a complete picture of the worker's fatigue exposure across the cycle.

8. Model Assumptions and Limitations

8.1 Assumptions

- **Population-level modelling:** The model represents a typical healthy working-age adult. Individual differences in chronotype, age, sleep disorders, medication use, and lifestyle are not modelled.
- **No environmental factors:** Workload intensity, noise, lighting, temperature, and other environmental variables that influence alertness are not incorporated.
- **Master schedule analysis:** The model evaluates the planned rotation design, not day-to-day variations such as overtime, shift swaps, call-ins, or unplanned absences.
- **Steady-state assumption:** Results represent the repeating cycle once the worker has adapted to the rotation pattern.

8.2 Limitations

- Sleep inertia (Process W) is omitted, as described in Section 2.1. For rotations involving short-notice callouts or very early starts where workers may be woken from deep sleep, this component may be material.
- Commute time is not modelled. Workers with long commutes will have reduced sleep opportunities relative to model predictions.
- Caffeine and other countermeasures are not incorporated. The model represents unmedicated fatigue.
- Adaptation to night work is not modelled. Some research suggests partial circadian adaptation after several consecutive night shifts; the model maintains a fixed circadian rhythm.

9. Scientific References

- Åkerstedt, T. & Folkard, S. (1997). The three-process model of alertness and its extension to performance, sleep latency, and sleep length. *Chronobiology International*, 14(2), 115–123.
- Åkerstedt, T. (2003). Shift work and disturbed sleep/wakefulness. *Occupational Medicine*, 53(2), 89–94.
- Åkerstedt, T., Folkard, S. & Portin, C. (2004). Predictions from the three-process model of alertness. *Aviation, Space, and Environmental Medicine*, 75(3), A75–A83.
- Folkard, S. & Tucker, P. (2003). Shift work, safety and productivity. *Occupational Medicine*, 53(2), 95–101.
- Folkard, S., Åkerstedt, T. & Spencer, M.B. (2005). Towards a model for the prediction of alertness and/or fatigue on different sleep/wake schedules. *Human Factors and Ergonomics Society Annual Meeting*.
- Dawson, D. & Reid, K. (1997). Fatigue, alcohol and performance impairment. *Nature*, 388, 235.
- Van Dongen, H.P.A. & Dinges, D.F. (2005). Sleep, circadian rhythms, and psychomotor vigilance. *Clinics in Sports Medicine*, 24(2), 237–249.
- Sallinen, M. & Kecklund, G. (2010). Shift work, sleep, and sleepiness — differences between shift schedules and systems. *Scandinavian Journal of Work, Environment & Health*, 36(2), 121–133.

10. Document Control

Version 1.0 — Initial release: Three-Process Model implementation with calibrated parameters.

Version 1.1 — First-night-in-run enhancement: Added physiological modelling for transitional night shifts. Workers entering their first night shift after day shifts or rest days are modelled as awake from 06:00, reflecting the absence of pre-shift napping when transitioning from a daytime schedule. This produces appropriately elevated fatigue scores for the highest-risk shift in any night block.



Rotation Analytics

A division of Visser Ventures Corp.

PO Box 2234, Crossfield, Alberta, Canada

hello@rotationanalytics.ca | (403) 506-3636

rotationanalytics.ca

This document is provided for informational purposes. It describes the mathematical methodology used by Rotation Analytics in its Fatigue Risk Analysis service. This document does not constitute medical, safety, or legal advice.

This model is provided for occupational fatigue risk assessment of rotation designs. It does not constitute medical advice and should not be used as the sole basis for safety-critical decisions.

© 2026 Rotation Analytics. All rights reserved.